COMPARISON OF THE MATRIX TECHNIQUES FOR MODELING STRESS WAVES IN MULTILAYERED MEDIA

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Abstract

This paper describes the development of a general purpose model for wave propagation in Cartesian system. When isotropic materials are used, the model can account for elastic and visco-elastic isotropic materials, single or multi-layered structures, and free or leaky systems. Model determines what resonances can exist in order to satisfy the boundary conditions and the bulk wave propagation characteristics in each of the layers. These resonances control how ultrasonic waves will be guided in the system and what properties each of these waves will have.

Key words: guided waves, dispersive curves

1 Introduction

Research into the use of ultrasonics for nondestructive inspection frequently involves the study of the interaction of sound with multilayered plate structures. Inspection methods for such tasks can be considered broadly in two groups: response methods in which the reflection and transmission characteristics of the plate are examined, and modal method which address the plate wave propagation properties of the system. The development of inspection techniques based on either approach requires the study of complicated wave mechanics and relies strongly on the use of predictive modeling tools to enable the best inspection strategies to be identified and their sensitivities to be evaluated.

The modeling tools may be developed from matrix formulations which describe elastic waves in layered media with arbitrary numbers of layers. The matrix formulations have featured in a large number of publications. Indeed there are currently two quite different approaches and many variants which are in accepted use. The purpose of this paper is to compare the main developments of the techniques and their implementation in response and modal models.

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2 Field Equations for Plane Waves in Flat Isotropic Elastic Layers

The field equations for the displacements and stresses in a flat isotropic solid layer may be expressed as the superposition of the fields of four bulk waves within the layer.

The displacement equations of motion in the vector form:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} \tag{1}$$

where $\mathbf{u}$ are displacements in the Cartesian coordinate system $\mathbf{x}$, t is time, $\rho$is density, λ and μ are Lamé's elastic stiffness constants.

This equation cannot be integrated directly. A convenient way of presenting the solutions in vector form is by the Helmholtz method, in which longitudinal waves (L) are described by a scalar function ϕ and shear waves (S) vector function $\mathbf{H}$ whose direction is normal to both the direction of wave propagation and the direction of particle motion:

$$\phi = A_L e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \qquad |\mathbf{H}| = A_S e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{2}$$

Here A_L and A_S are the longitudinal and shear wave amplitudes, $\mathbf{k}$ is the wavenumber vector and $\omega$is the angular frequency.

Fig. 1 illustrated the coordinate system which will be used for the plate. For plane strain there is no variation of any quantity in the x_3 direction. Furthermore, the model is restricted to waves whose particle motion is entirely in the plane $u_3 = 0$.

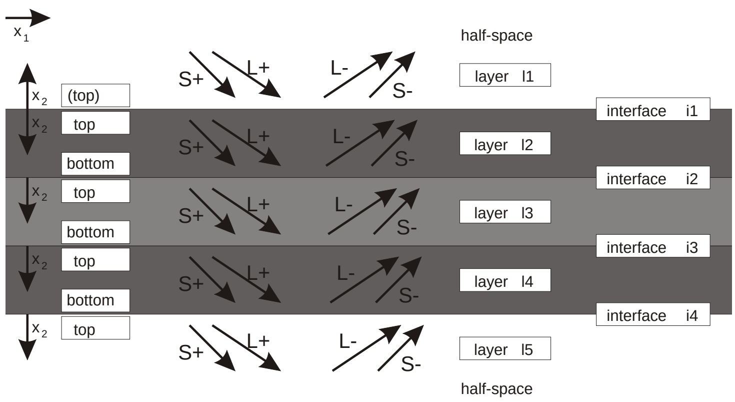

Figure 1: Labeling system for the multilayered plate.

The development of a model for wave motion in multilayered plates is achieved by the superposition of longitudinal and shear bulk waves and the imposition of boundary conditions at the interfaces between the layers. At each interface, it is sufficient to assume eight waves: longitudinal and shear waves arriving from "above" the interface and leaving "below" the interface (L+, S+) and, similarly, longitudinal and shear waves arriving from below the interface and leaving above the interface (L-, S-). Snell's law requires that for interaction of the waves they must all share the same frequency and spatial properties in the x_1 direction at each interface. It follows that all displacement and stress equations have the same ω and the same $k_1 = \xi$ component of wavenumber, being the projection of the wavenumber of the bulk wave onto the interface. All field equations therefore contain the following factor, F, which is an invariant of the system: $F = e^{i(\xi x_1 - \omega t)}$.

The displacements and stresses at any location in a layer may therefore be found by summing the contributions due to the four wave components in the layer. The field quantities of interest are those which must be continuous at the interfaces: the two displacement components u_1 and u_2, the normal stress σ_{22} and the shear stress σ_{12}. Making the substitutions for convenience, $\zeta_1 = \sqrt{\omega^2/c_1^2 - \xi^2}$, $\zeta_2 = \sqrt{\omega^2/c_2^2 - \xi^2}$, $g_{\zeta_1} = e^{i\zeta_1 x_2}$, $g_{\zeta_2} = e^{i\zeta_2 x_2}$, and omitting the common factor, F, the field quantities in a layer are thus expressed by the matrix equation:

$$\begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \xi g_{\zeta_1} & \frac{\xi}{g_{\zeta_1}} & \zeta_2 g_{\zeta_2} & -\frac{\zeta_2}{g_{\zeta_2}} \\ \zeta_1 g_{\zeta_1} & -\frac{\zeta_1}{g_{\zeta_1}} & -\xi g_{\zeta_2} & -\frac{\xi}{g_{\zeta_2}} \\ i\rho\left(\omega^2 - 2c_2^2\xi^2\right) g_{\zeta_1} & \frac{i\rho\left(\omega^2 - 2c_2^2\xi^2\right)}{g_{\zeta_1}} & -2i\rho\xi c_2^2\zeta_2 g_{\zeta_2} & \frac{2i\rho\xi c_2^2\zeta_2}{g_{\zeta_2}} \\ 2i\rho\xi c_2^2\zeta_1 g_{\zeta_1} & \frac{-2i\rho\xi c_2^2\zeta_1}{g_{\zeta_1}} & i\rho\left(\omega^2 - 2c_2^2\xi^2\right) g_{\zeta_2} & \frac{i\rho\left(\omega^2 - 2c_2^2\xi^2\right)}{g_{\zeta_2}} \end{bmatrix} \begin{bmatrix} A_{L+} \\ A_{L-} \\ A_{S+} \\ A_{S-} \end{bmatrix} \tag{3}$$

Matrix in (3) is the field marix, describing the relationship between the wave amplitudes and the displacements and stresses at any location in any layer. Its coefficients depend on the through-thickness position in the plate (x_2), the material properties of the layer at the position (ρ, c_1 a c_2), the frequency (ω)¸and the invariant plate wavenumber ($\xi = k_1$). **The origin of the x_2 coordinate may be placed arbitrary and may even be different for each layer because phase differences between layers can be accounted for by the phase of the complex wave number.** The field matrix will be abbreviated here to [**D**].

3 The Transfer Matrix Method

The Transfer Matrix method works by condensing the multilayered system into a set of four equations relating the boundary conditions at the first interface to the boundary conditions at the last interface (Thomson [3]). In the process, the equations for the intermediate interfaces are eliminated so that the fields in all of the layers of the plate are described solely in terms of the external boundary conditions.

A five layer system is illustrated as an example (Fig. 1), consisting of a three layer plate with two half-spaces. The layers of the system are labeled $l1$ to $l5$, and the interfaces, $i1$ to $i4$. Each layer has its own x_2 origin, defined as the location of its top interface, except for the first layer ($l1$) which has its origin at its interface with ($l2$).

Assume that the displacements and stresses are known at the first interface, ($i1$). The amplitudes of the four waves at the top of layer $l2$ can now be found by inverting the matrix $[D]$:

$$\begin{bmatrix} A_{(L+)} \\ A_{(L-)} \\ A_{(S+)} \\ A_{(S-)} \end{bmatrix}_{l2} = [D]^{-1}_{l2,\mathrm{top}} \begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{l2,\mathrm{top}} \tag{4}$$

At the second interface, $(i2)$, the displacements and stresses at the bottom of the layer can be found from the wave amplitudes in layer $l2$:

$$\begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{l2,\text{bottom}} = [D]_{l2,\text{bottom}} \cdot [D]^{-1}_{l2,\text{top}} \begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{l2,\text{top}} \tag{5}$$

The matrix product in this equation now relates the displacements and stresses between the top and bottom surfaces of a single layer and may be reffered to as the layer matrix, $[L]$, which for layer $l2$ is:

$$[L]_{l2} = [D]_{l2\text{bottom}} \cdot [D]^{-1}_{l2,\text{top}} \tag{6}$$

The inverted $[D]$ matrix may be expressed explicitly [2] and therefore it is possible to write out the coefficients of the $[L]$ matrix. The displacements and stresses must be continuous across a "welded" interface between two layers. Therefore

$$\begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{l3,\text{top}} = \begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{l2,\text{bottom}} = [L]_{l2} \begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{l2,\text{top}} \tag{7}$$

Clearly this process can be continued layer by layer for all subsequent layers¸resulting in the equation:

$$\begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{ln,\text{top}} = [S] \begin{bmatrix} u_1 \\ u_2 \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_{l2,\text{top}} \tag{8}$$

where n is the last layer and $[S]$ is the system matrix consisting of the matrix product of the layer matrices.

4 The Global Matrix Method

In 1964 Knopoff published a fundamentally different matrix formulation for multilayered media [1]. The approach with the global matrix method is to assemble directly a single matrix which represents the complete system. The system matrix consists of $4\,(n-1)$ equations, where n is the total number of layers. The equations are based, in sets of four, on satisfying the boundary conditions at each interface. Thus no assumption is made a priori about any interdependence between the sets of equations for each interface. This does not mean that the interfaces are completly independent, bacause the equations at an interface are influenced by the arrival of waves from the neighboring interfaces. However, as the frequency-thickness product is increased, the influence of an inhomogeneous wave travelling along one interface on the displacements and stresses at the next interface simply reduces. The method therefore remains perfectly stable for any frequency-thickness produkt.

Consider a single interface, for example the second interface (*i2*) in Fig. 1. Utilizing (3), the displacements and stresses at the interface can be expressed as a function of the amplitudes of the waves at the top of the third layer (*l3*). They may also be expressed as a function of the amplitudes of the waves at the bottom of the second layer (*l2*). For continuity of displacements and stresses at the interface, both expressions should give equal results. Therefore

$$[[D_{2b}]\,[-D_{3t}]]\begin{bmatrix} A_{(L+)2} \\ A_{(L-)2} \\ A_{(S+)2} \\ A_{(S-)2} \\ A_{(L+)3} \\ A_{(L-)3} \\ A_{(S+)3} \\ A_{(S-)3} \end{bmatrix} = 0 \tag{9}$$

where the subscripts 2 and 3 refer to layers *l2* and *l3* and t and b to the top and bottom of each layer. This equation describes the interaction at interface *i2* of the waves in the adjoining layers *l2* and *l3*.

Before proceeding, a modification is made to the origins of he bulk waves, which will affect the field (3). **Instead of defining the origin for all of the waves in a layer to be the top of the layer, the origin of all waves is defined to be at their entry to the layer.** Thus downward traveling waves have their origin at the top of the layer and upward travelling waves have their origin at the bottom of the layer. No change is made for the half-spaces. With this modification, and referring to (3), the $[D]$ matrices for the top and bottom of a layer can be expressed, respectively, as:

$$[D_t] = \begin{bmatrix} k_1 & k_1 g_\alpha & C_\beta & -C_\beta g_\beta \\ C_\alpha & -C_\alpha g_\alpha & -k_1 & -k_1 g_\beta \\ i\rho B & i\rho B g_\alpha & -2i\rho k_1 \beta^2 C_\beta & 2i\rho k_1 \beta^2 C_\beta g_\beta \\ 2i\rho k_1 \beta^2 C_\alpha & -2i\rho k_1 \beta^2 C_\alpha g_\alpha & i\rho B & i\rho B g_\beta \end{bmatrix}$$

$$[D_b] = \begin{bmatrix} k_1 g_\alpha & k_1 & C_\beta g_\beta & -C_\beta \\ C_\alpha g_\alpha & -C_\alpha & -k_1 g_\beta & -k_1 \\ i\rho B g_\alpha & i\rho B & -2i\rho k_1 \beta^2 C_\beta g_\beta & 2i\rho k_1 \beta^2 C_\beta \\ 2i\rho k_1 \beta^2 C_\alpha g_\alpha & -2i\rho k_1 \beta^2 C_\alpha & i\rho B g_\beta & i\rho B \end{bmatrix} \tag{10}$$

A similar equation to (9) can now be written for the interface *i3* and simply added to the global matrix, and similarly for all interfaces, resulting in a matrix of $4\,(n-1)$ equations and $4n$ unknowns. In our case the matrix equation is:

$$\begin{bmatrix} [D_{1b}] & [-D_{2t}] & & & \\ & [D_{2b}] & [-D_{3t}] & & \\ & & [D_{3b}] & [-D_{4t}] & \\ & & & [D_{4b}] & [-D_{5t}] \end{bmatrix} \cdot \begin{bmatrix} [A_1] \\ [A_2] \\ [A_3] \\ [A_4] \\ [A_5] \end{bmatrix} = [0] \tag{11}$$

where the wave amplitudes in each layer, $A_{(L+)}$, $A_{(L-)}$, $A_{(S+)}$, $A_{(S-)}$, have been abbreviated simply to a layer wave vector $[A]$. Four of the wave amplitudes in (11) must now be identified as knowns and their coefficients in the equations moved to the right hand side. For ultrasonics applications it is conveniet to choose the incoming waves in the two half-spaces, $A_{(L+)1}$, $A_{(S+)1}$, $A_{(L-)5}$ and $A_{(S-)5}$, as the knowns, giving:

$$\begin{bmatrix} [D_{1b}^-] & [-D_{2t}] & & & \\ & [D_{2b}] & [-D_{3t}] & & \\ & & [D_{3b}] & [-D_{4t}] & \\ & & & [D_{4b}] & [-D_{5t}^+] \end{bmatrix} \begin{bmatrix} [A_1^-] \\ [A_2] \\ [A_3] \\ [A_4] \\ [A_5^+] \end{bmatrix} = \begin{bmatrix} [-D_{1b}^+] & \\ & \\ & \\ & [D_{5t}^-] \end{bmatrix} \begin{bmatrix} [A_1^+] \\ [0] \\ [0] \\ [0] \\ [A_5^-] \end{bmatrix} \tag{12}$$

where the superscripts + and - denote those parts of the matrices or vectors corresponding to + and - waves, respectively. Thus the vectors $[A^+]$ and $[A^-]$ each consist of half of the vector $[A]$ and the matrices $[D^+]$ and $[D^-]$ are four-by-two partitions of the matrix $[D]$.

The system matrix on the left hand side of (12) and the sparse matrix on the right side are both square and of dimension $4\,(n-1)$. If the wave amplitudes for the incoming waves are knows then the right hand side may be evaluated immediately.

5 Conclusion

The Global Matrix method has the advantages that it is robust and that the same matrix may be used for all categories of solution, whether response or modal, vacuum or solid half-spaces, real or complex plate wavenumber. The disadvantage is that the global matrix may be large and therefore the solution may be relatively slow.

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