COMPUTATIONAL MECHANICS 2005
21st conference with international participation
Hrad Nečtiny
November 7 - 9, 2005

ANALYTICAL, NUMERICAL AND EXPERIMENTAL INVESTIGATION OF NON-STATIONARY STATE OF STRESS IN A THIN VISCOELASTIC PLATE

Vítězslav ADÁMEK [1], **František VALEŠ** [2]

Abstract: *Presented work deals with solution of non-stationary state of stress in a thin viscoelastic plate. The first section is focused on analytical and numerical solution of the problem of transverse non-stationary loaded viscoelastic plate. The analytical solution stated was derived using the theory of thin plates with so called Timoshenko-Mindlin correction and it serves for the validation of numerical model. Afterwards, the real problem of the impact of small glass ball on the surface of a thin viscoelastic plate is solved in the second section. Concretely, numerical and experimental results are presented and discussed. Their comparison showed the necessity of material model parameters modification. This step led to better compliance between numerical and experimental results.*

Keywords: *viscoelastic, thin plate, solution, non-stationary, waves*

1. INTRODUCTION

The reason for the choice of the topic of this work is the fact that conventional materials are very often substituted by new modern materials that often exhibit viscoelastic behaviour (dissipation of energy). We meet bodies with these properties not only in industrial applications, but also in daily life and in many different forms (e.g. plastics, human tissue, wood, etc.). Plastics, especially polymers, represent the greatest set of such materials. They are usually utilized in industry directly or in the form of composite matrices whereby they significantly influence composites behaviour. With respect to their specific mechanical properties, viscoelastic materials are often used in structures subjected to dynamic loading with stationary or non-stationary character (e.g. impacts). Therefore it is important to be able to describe and predict the state of stress in components under such working conditions.

This work concerns non-stationary phenomena in viscoelastic solids. This part of continuum mechanics is studied by many authors already since the beginning of the 20th century. Detailed overview of works that concerned dynamical properties and wave phenomena in viscoelastic materials and that were published until the 60s of the 20th century gave Kolsky (1958). In his work one can find mentions of solutions of stress waves propagation in 1D and 3D continuum with both linear and non-linear viscoelastic properties of such authors as Havelock, Lee, Kanter, Morrison, Kolsky, Charles, Davies, O'neill, etc. From the large number of works that appeared at a later time one can mention the work Zhao and Gary (1995) where the authors generalized the solution of longitudinal waves propagation in a thick elastic bars derived by Pochhammer in 1876 and Chree in 1889 to viscoelastic problems. All mentioned authors deals with wave problems mainly from analytical or experimental point of view.

[1] Ing. Vítězslav Adámek, Ph. D., Západočeská univerzita v Plzni, Univerzitní 22, 306 14 Plzeň, tel.: +420 377632335, e-mail: vadamek@kme.zcu.cz

[2] Ing. František Valeš, CSc., Ústav termomechaniky AV ČR, Veleslavínova 11, 301 14 Plzeň, tel.: +420 377236415, e-mail: vales@cdm.it.cas.cz

Nowadays, when computer equipment and numerical methods are at high level, most of real problems that are more complicated than the theoretical ones are solved numerically. Many works that deal with numerical solutions of waves propagation in viscoelastic solids and their application in transportation engineering, medicine and geomechanics can be found. The finite difference method, the Lax-Wendroff scheme or FEM are usually used for space discretization in such cases. When the non-stationary wave problems are solved numerically, one has to pay attention if the method used is able to involve sharp fronts of waves propagated. In the case of FEM, the mostly used method, this fact is related to so called limit frequency of FE model. It means that waves containing frequencies higher than this value are not represented by the FE model correctly and they are distorted.

2. PROBLEM OF TRANSVERSE NON-STATIONARY LOADED PLATE

This section concerns analytical and numerical solution of non-stationary state of stress in an infinite thin plate of thickness h that is transverse loaded by uniformly distributed pressure on its upper face. The loading applied has nonzero constant amplitude σ_0 only in the circular area with finite radius R and it changes according to the Heaviside function in time. With respect to the axial symmetry of applied loading and to infinity of the plate, the problem is solved as axisymmetric one and cylindrical coordinates are used. The material of the plate is assumed to be homogenous linear viscoelastic and it is represented by standard linear viscoelastic solid (Zener model). Volume changes are not taken into account, the material is assumed incompressible, so its behaviour is represented only by relaxation function of shear modulus in the form $G(t) = G_0 + G_1 e^{-\frac{t}{t_\varepsilon}}$, where G_0 and G_1 represent the shear modulus of alone standing spring and the spring with dashpot in series, respectively. Time t_ε is the relaxation time of the branch with dashpot.

2.1 Analytical solution

The presented analytical solution was completely derived in Adámek (2004). Since time history plots and path plots of radial strain component ε_r will be observed in this work, corresponding analytical expression will be only stated. As mentioned above, the theory of thin plates was used for the derivation of motion equations. To achieve better compliance with three dimensional theory of continuum, so called Timoshenko-Mindlin correction was used. It means that the effect of shear and the rotary inertia of an element are taken into account as well. The resulted system of two dependent partial-integrodifferential equations of the second order was solved by Hankel and Laplace integral transformations. Under the assumption of zero boundary and initial conditions, the function describing spatio-temporal distribution of strain component ε_r can be written in the form

$$\varepsilon_r(r,t) = \sigma_0 R z \int_0^\infty J_1(\gamma R)\left(\gamma J_0(\gamma r) - \frac{1}{r} J_1(\gamma r)\right)\left(\sum_{n=0}^{6}\left[\frac{P_2 e^{pt}}{\frac{\partial}{\partial p}(pP_6)}\right]_{p=p_n}\right) d\gamma\,, \tag{1}$$

where t is time, $p \in C$ is variable of Laplace transformation, $\gamma \in R$ is variable of Hankel transformation, r and z correspond to radial distance from the axis of symmetry and transverse distance from the middle surface of the plate, respectively. Functions J_0 and J_1 represent the Bessel's functions of the first kind and the zeroth and the first order. The complex polynomials $P_2(\gamma, p)$ and $P_6(\gamma, p)$ of the second and the sixth order involve material and geometric characteristics of the problem solved and their exact form can be found in Adámek (2004). The sum in (1) implies from the Cauchy theorem and the theorem of residue and it substitutes the integral of inverse Laplace transformation. The analytical solution (1) was then evaluated using numerical and symbolical functions of system Matlab and Maple. The Simpson's rule was used for the evaluation of the integral of inverse Hankel transformation. Examples of resulted time history plots and path plots of radial strain component ε_r are presented in section 2.3.

2.2 Numerical solution

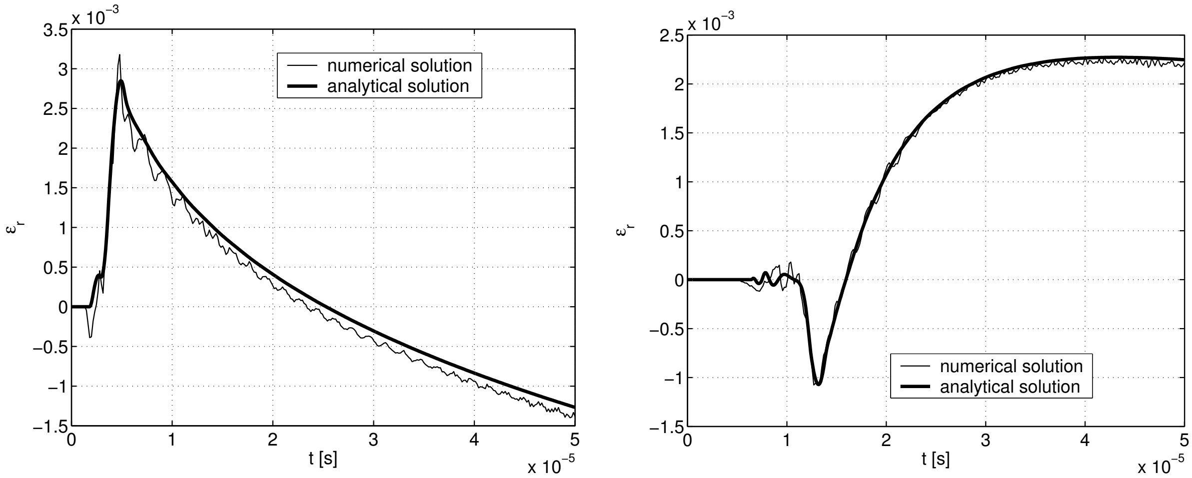
Numerical simulation of the problem solved was performed in FE software MSC.MARC/Mentat. This system supports solution of axisymmetric problems such that whole geometry of numerical model consisted of one half of transverse cross-section of the plate. It led to significant reduction of CPU time and RAM requierements. Finite radius 75 mm of the plate and its thickness 2 mm were chosen for numerical simulation. The regular FE mesh of the model consisted of quadrangular 4-noded isoparametric elements with linear base functions. Their basic size was chosen 0.5×0.5 mm. This mesh was twice refined in direction to applied loading. The first refinement was realized at radius 40 mm, the second one at radius 35 mm, i.e. the elements had size 0.125×0.125 mm round about the external pressure where the results were observed.

Total time of numerical simulation $5 \cdot 10^{-5}$ s was divided into 4000 increments, i.e. time interval $1.25 \cdot 10^{-8}$ s corresponded to one integration step. The size of integration step was chosen according to the size of the smallest element and to the phase velocity of waves propagated. Newmark algorithm was used for integration in time domain. Boundary conditions of the model were defined so, that they represented the clamped circumference of the plate and the uniformly distributed pressure loading. The pressure was applied to the elements inside the circle with radius 1 mm on the upper plate surface and its amplitude changes from 0 MPa to 50 MPa according to the Heaviside time step function.

2.3 Results comparison

Analytical solution (1) was evaluated for geometry and external loading specified in previous subsection. Material parameters of the plate were chosen for both analytical and numerical solutions as follows: $G_0 = 1.2869 \cdot 10^9$ Pa, $G_1 = 2.82786 \cdot 10^8$ Pa, $t_\varepsilon = 3.68932 \cdot 10^{-5}$ s, $\rho = 1140$ kgm^{-3}, $\mu = 0.4$. These values were used in Bussac et al. (2002) where the authors investigated waves propagation in nylon bars.

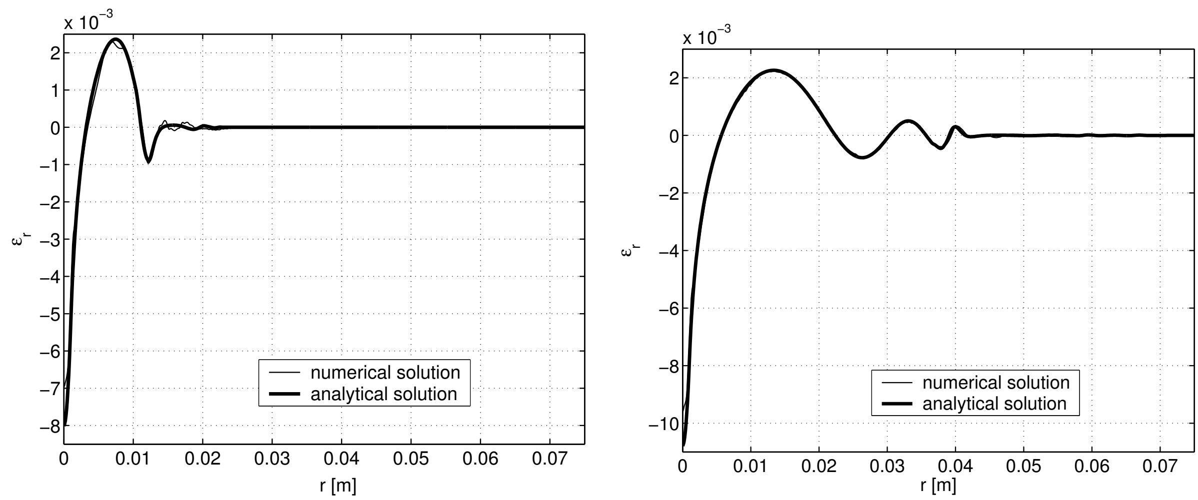
Comparison of results obtained is performed in fig. 1 and in fig. 2 where time history plots of ε_r for $r = 5$ mm and $r = 15$ mm and the distribution of ε_r along plate radius at time $t = 10^{-5}$ s and $t = 33 \cdot 10^{-6}$ s are depicted, respectively. Thinner lines correspond to numerical solution and thicker ones to analytical results. It is clear from these figures that the analytical solution implicit in the approximate theory of thin plates and the numerical solution implicit in three dimensional theory are in good concordance. Naturally, the compliance is better at longer time and for larger distances from excitation because the high-frequency components of the waves are already damped due to their higher attenuation coefficient than



(a) History plot of ε_r for $r = 5$ mm.

(b) History plot of ε_r for $r = 15$ mm.

Fig. 1. Comparison of results obtained - history plots of ε_r.



(a) Path plot of ε_r at $t = 10^{-5}$ s.

(b) Path plot of ε_r at $t = 33 \cdot 10^{-6}$ s.

Fig. 2. Comparison of results obtained - path plots of ε_r.

in the case of low-frequency components (for more detail see Adámek (2004)). Based on presented results one can say that the parameters of numerical model (elements size, base functions, integration step, etc.) were chosen correctly and the model created can be used for simulation of similar non-stationary wave phenomena in solids. With respect to the fact that the proportional representation of high-frequency components in the Fourier spectrum of contact force is smaller than in the case of excitation in the form of Heaviside function, the model parameters can be used in the following section where numerical and experimental investigation of non-stationary state of stress in a thin viscoelastic plate caused by the impact of ball on its surface is presented.

3. PROBLEM OF TRANSVERSE IMPACT ON PLATE

As mentioned above, the contact impact problem will be solved in this section. Concretely, the real problem of the impact of a small glass ball with diameter 5 mm perpendicularly to the plate surface will be numerically and experimentally investigated. The plate specimen of size 150×150 mm was made from polyamide 6 (nylon 6) that is produced with trade name Tecamid 6 by Ensinger company. This material was chosen for its significant viscoelastic properties and for its wide utilization in most branches of engineering industry. It has good sliding properties, good chemical resistance, it is very abrasion resistant, very tough, rigid, electrically insulating and easily machined and that is why it is used for the producing of gear wheels, friction strips, piston guides, impact and damping plates, friction bearings, conveyor screws, etc.

3.1 Numerical solution

The problem was solved again as axisymmetric one. The model from subsection 2.2 was the base for numerical model of this problem. The geometry and the mesh of the plate were retained, only boundary condition representing external loading was substituted by contact body in the form of one half of cross-section of impacting ball with radius specified. With respect to the fact that the wave phenomena were explored in the plate, the mesh of the ball contained quadrangular elements with approximate edge length 0.25 mm, i.e. roughly twice greater than the mesh of the plate in area of interest.

Parameters of material model representing viscoelastic properties of the thin plate were identical to them used in previous section. Material properties of the glass ball were modeled as elastic with Young modulus $E = 7.5 \cdot 10^{10}$ Pa, density $\rho = 2580$ kgm^{-3}, Poisson's ratio

$\mu = 0.16$. Total time of the analysis 10^{-4} s was divided into 8000 identical steps so time $1.25 \cdot 10^{-8}$ s corresponded to one integration step of Newmark method.

3.2 Experiment

The external company Lenam, s.r.o. from Liberec that has the equipment necessary for experimental investigation of non-stationary phenomena in solids was asked for realization of experiment. Authors's workplaces have long-standing cooperation with workers of this company who are experts in this branch.

The air cannon that is able to shoot balls with diameter 5 mm with predefined velocity was used for the realization of impact. The plate polyamide specimen was fixed against this cannon by a gripper and four strain gauges were glued to its surface. These semiconductor strain gauges of type AP120-1.5-12 with length 1 mm and resistance $120\,\Omega$ were glued 10 mm, 15 mm, 20 mm and 35 mm far from the point of ball impact in radial direction to be able to measure radial strain component ε_r. Further, the laboratory power supply TSZ 75 and the four-channel oscilloscope TDS 2014 Tektronix with sampling rate 10 MHz were used. Actual arrangement of experimental equipment is depicted in fig. 3.

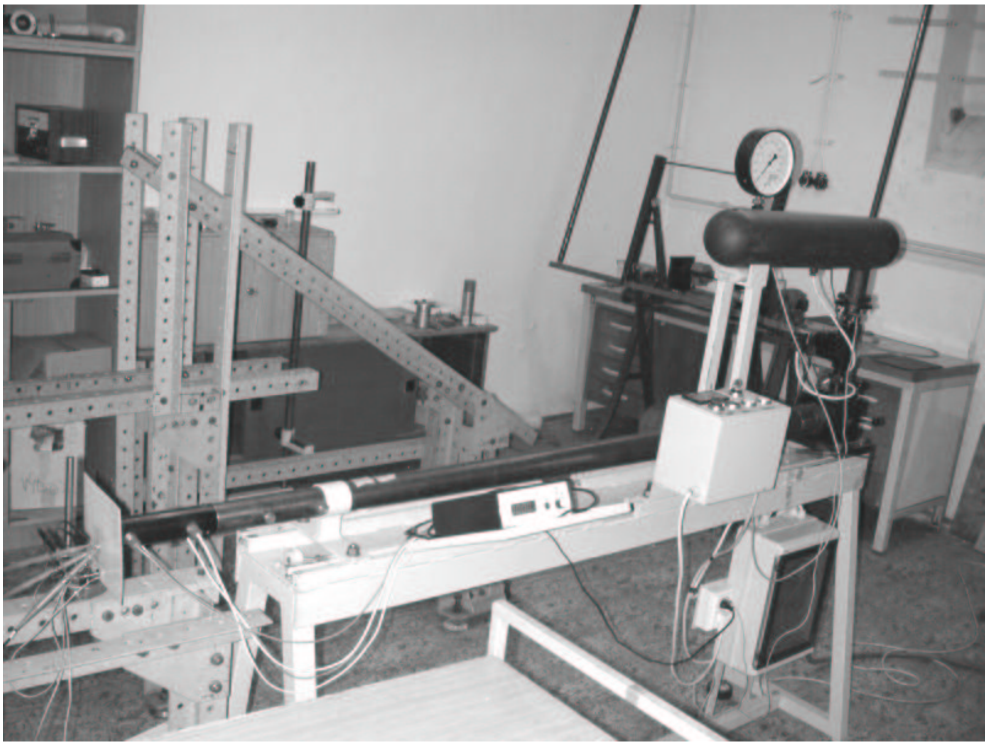
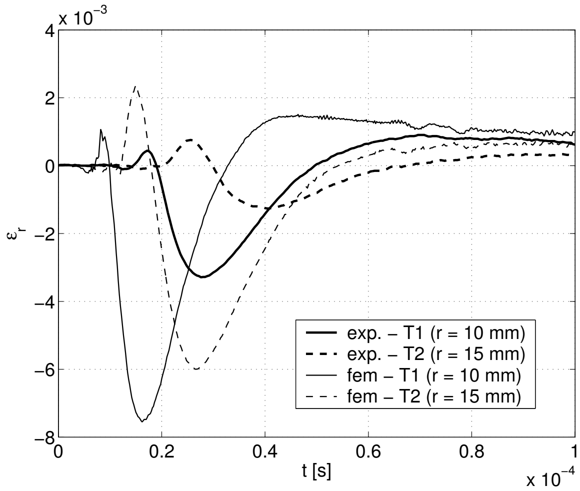


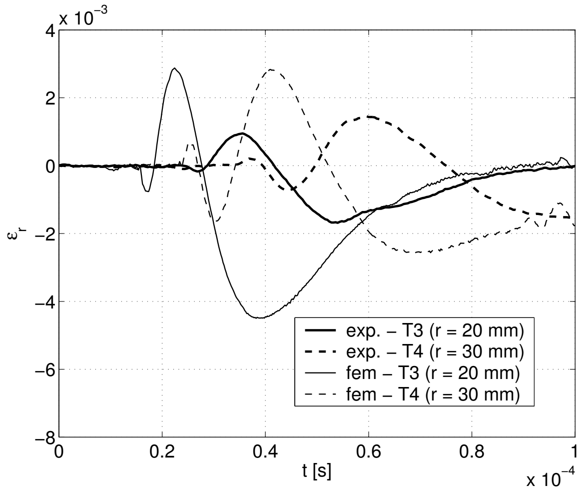
Fig. 3. Arrangement of experimental equipment.

3.3 Results and their comparison

The first experiment that was three times repeated to ensure its credibility was performed for the impact velocity $40\,\mathrm{ms}^{-1}$. With respect to the fact that exact regulating of ball velocity is not possible in the case of air cannon, actual velocities were $39.7\,\mathrm{ms}^{-1}$, $40.3\,\mathrm{ms}^{-1}$ and $43.5\,\mathrm{ms}^{-1}$. The signals from strain gauges were recorded for $6 \cdot 10^{-4}$ s. After realization of corresponding numerical simulations and after comparison of numerical and experimental results within time intervals in which the influence of different boundary conditions in experiment and numerical model did not approved, significant deviations of numerical and experimental results were observed. These deviations persisted after the averaging of numerical results over the length of strain gauge as well. Mentioned deviations are presented in fig. 4 where numerical (curves *fem*) and experimental (curves *exp.*) results for impact velocity $39.7\,\mathrm{ms}^{-1}$ are



(a) History plots of ε_r for $r = \{10, 15\}$ mm.



(b) History plots of ε_r for $r = \{20, 30\}$ mm.

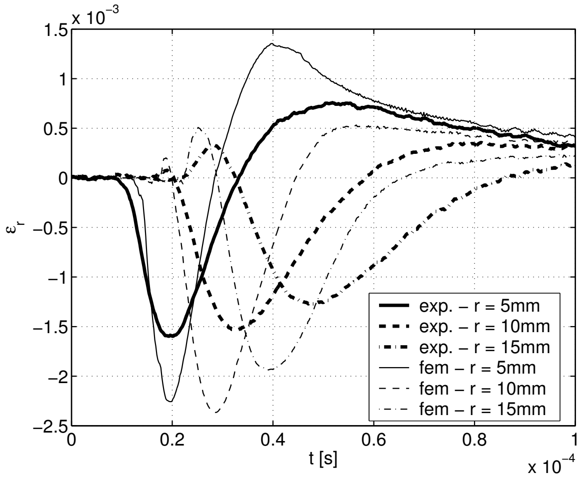
Fig. 4. Comparison of numerical and experimental results for impact velocity $39.7\,\mathrm{ms}^{-1}$.

depicted. Notations T1, T2, T3 and T4 subsequently correspond to strain gauges mounted $10\,$mm, $15\,$mm, $20\,$mm and $35\,$mm far from impact point. It is obvious from results stated that the amplitudes of numerically determined radial strain are more than twice greater than corresponding experimental ones. The time shift of both results is also evident, but it is the consequence of impossible exact setting of time $t = 0\,$s on experimental equipment.

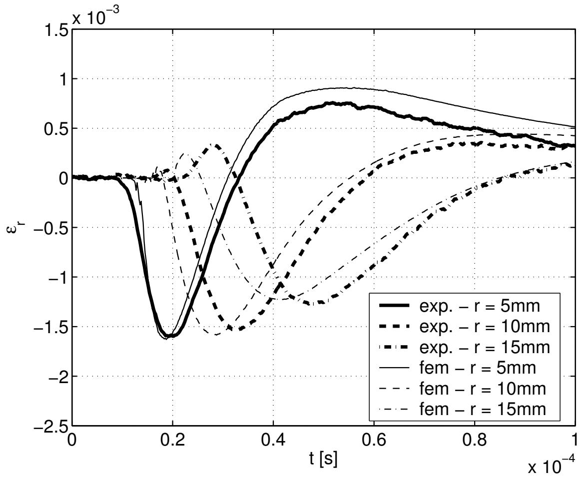
The existence of plastic deformations at impact point and its surroundings that can be the cause of energy dissipation and sequential decrease of wave amplitudes was firstly designated as the reason of problems mentioned. Additional experiments with lower impact velocity were performed to prove this hypothesis, concretely nine experiments with impact velocities $\{13.35, 11.16, 11.43, 19.4, 20.83, 20.96, 29.5, 30.9, 30.4\}\,\mathrm{ms}^{-1}$ were executed. Since the strain gauge T4 was damaged during the experiment with impact velocity $43.5\,\mathrm{ms}^{-1}$, only strain gauges T1, T2 and T3 were used. The comparison of new results obtained showed that the decrease of impact velocity did not lead to significant reduction of deviations between numerical and experimental results.

Based on such continuing problems, correctness of material model parameters used was revalued. As mentioned in subsection 2.3, the parameters were used in Bussac et al. (2002) for investigation of non-stationary waves in nylon bars. When we take into consideration that the set of polymers called nylon is very extensive, it is possible that the material parameters of standard viscoelastic solid are different for Tecamid 6. Since the values of model parameters required were not found for this material in available literature, the analysis of influence of each material parameter on numerical results of radial strain ε_r was performed to find their approximate values. The set of material parameters that can be varied reduces to coefficient of viscosity η and Young moduli E_0 and E_1 of alone standing spring and the spring with dashpot in series in the model of viscoelastic standard solid, respectively. Remaining parameters such as density and Poisson's ratio are determined by manufacturer of Tecamid 6. Starting values of searched parameters were chosen according to Bussac et al. (2002), i.e. $E_0 = 3.6032 \cdot 10^9\,$Pa, $E_1 = 7.918 \cdot 10^8\,$Pa and $\eta = 29212\,\mathrm{Pas}^{-1}$.

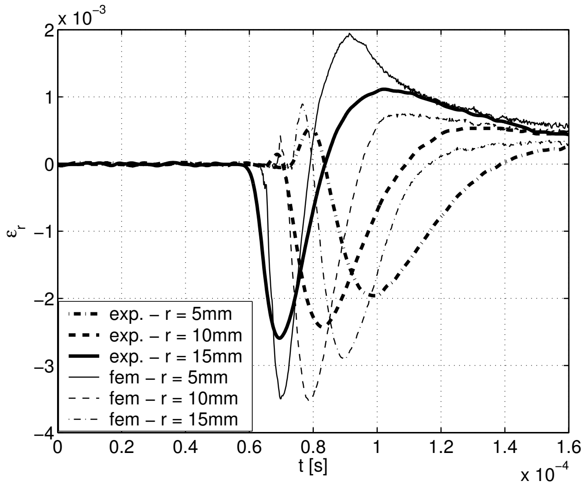
More than 60 numerical simulations for different values of η, E_0 and E_1 were performed to approach experimentally determined results. After the analysis of numerical results obtained, the approximate values of required parameters were found and they are: $E_0 = 2.7 \cdot 10^9\,$Pa, $E_1 = 5 \cdot 10^9\,$Pa and $\eta = 20000\,\mathrm{Pas}^{-1}$. Applicability of these parameters for modelling of Tecamid 6 can be examined from following fig. 5 where comparison of numerical and experimental results for impact velocities $13.35\,\mathrm{ms}^{-1}$, $19.4\,\mathrm{ms}^{-1}$ and $29.5\,\mathrm{ms}^{-1}$ for origin values of parameters and for new ones is stated. It is evident from fig. 5(a) - 5(f) that new material pa-



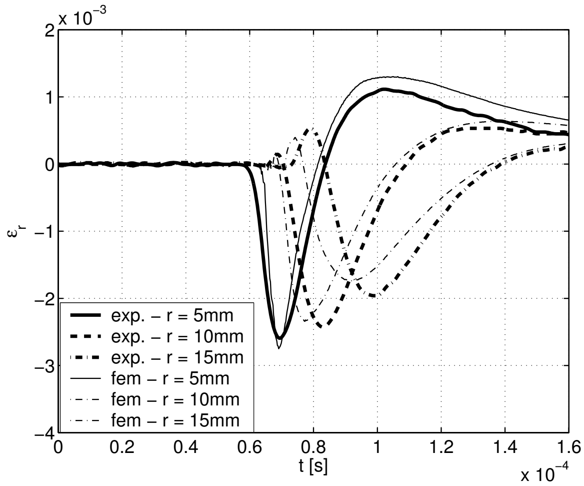
(a) Impact velocity $13.35\,\mathrm{ms}^{-1}$ - original model.



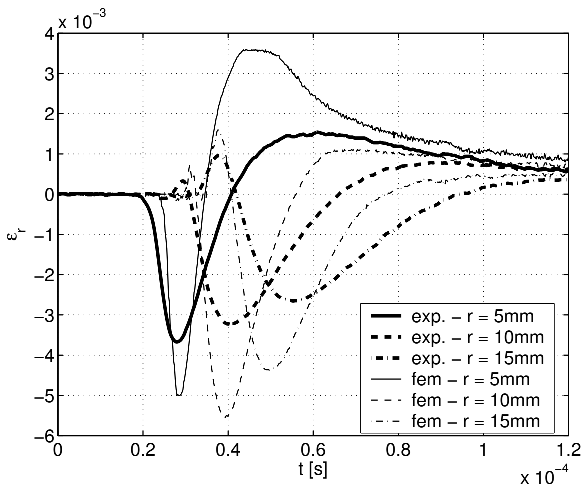
(b) Impact velocity $13.35\,\mathrm{ms}^{-1}$ - new model.



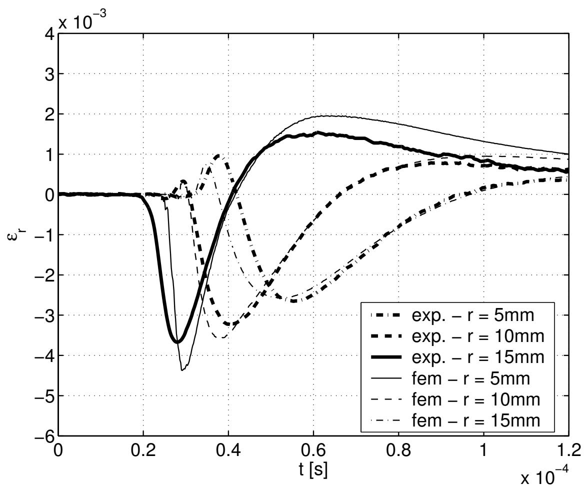
(c) Impact velocity $19.4\,\mathrm{ms}^{-1}$ - original model.



(d) Impact velocity $19.4\,\mathrm{ms}^{-1}$ - new model.



(e) Impact velocity $29.5\,\mathrm{ms}^{-1}$ - original model.



(f) Impact velocity $29.5\,\mathrm{ms}^{-1}$ - new model.

Fig. 5. Comparison of results before modification of material parameters and after it.

rameters significantly improve correspondence between numerical and experimental results for impact velocity approximately up to $30\,\mathrm{ms}^{-1}$. For the velocities about $40\,\mathrm{ms}^{-1}$ the numerical results still very differ from experimental data. It is probably caused by the fact that the real problem solved can not be considered as linear upon such high velocities so that experimental results can not be compared to them obtained using linear numerical model. This hypothesis was confirmed by exploration of the dependence of maximal radial strain on impact velocity. This dependence was linear approximately up to $30\,\mathrm{ms}^{-1}$ and non-linearity took effect over this value. It stands to reason that the material model of standard linear viscoelastic solid with founded parameters can not exactly represent real material properties of polyamide plate but it can serve as their approximation.

4. CONCLUSION

In this work analytical, numerical and experimental investigation of non-stationary phenomena in a thin viscoelastic plate was performed. Firstly, the analytical solution was presented, evaluated using Matlab and Maple functions and then compared with results of numerical simulation (FEM). This procedure serves for verification of numerical model whose basic setting could be then used for solution of real impact of small glass ball on the surface of polyamide plate. Owing to the comparison of numerical solution with experimental results, the parameters of material model were modified so that the resulted numerical model can be used for simulation of such problems. Additionally, the range of applicability of mentioned model was specified.

Acknowledgement: The work has been supported by the research project MSM 4977751303 and by the project of ÚT AV ČR number IRPAVOZ 20760514.

REFERENCES

Adámek, V., 2004. *Analytical, numerical and experimental investigation of non-stationary state of stress in a thin viscoelastic plate*. Ph. D. thesis, West Bohemia University in Pilsen, Pilsen (in Czech).

Bussac, N. M., Collet, P., Gary, G., and Othman, R., 2002. An optimisation method for separating and rebuilding one-dimensional dispersive waves from multi-point measurements. Application to elastic or viscoelastic bars, *Journal of Mechanics and Physics of Solids*, Vol. 50. pp. 321-349.

Kolsky, H., 1958. The propagation of stress waves in viscoelastic solids. *Applied Mechanics Reviews*, Vol. 11. pp. 465-468.

Zhao, H., and Gary, G., 1995. A three dimensional analytical solution of the longitudinal wave propagation in an infinite linear viscoelastic cylindrical bar. Application to experimental techniques. *Journal of the Mechanics and Physics of solids*, Vol. 43. pp. 1335-1348.